

SOLUTION Exercise Sheet 1

Power Harvesting Factor

Consider a symmetrical, three-bladed ($B = 3$) wind turbine with rotor radius R . Assume a constant angular velocity Ω of the rotor and a uniform wind field with velocity $\mathbf{u}_\infty$ so that the dominant wind direction $\hat{\mathbf{i}}$ is along the turbine axis of rotation. We will also use a nondimensional spanwise position $\mu = r/R$ that is 0 at the blade root/rotor hub, and 1 at the blade tips.

1. What is the tip speed ratio λ of the turbine?

The tip speed ratio $\lambda = \frac{\Omega R}{u_\infty}$ is the ratio between the blade speed u_b at the tip and the freestream wind speed u_∞ .

(We'll use the abbreviation that $v = \|\mathbf{v}\|_2$ from here on.)

2. What is the local speed ratio λ_r at some spanwise location μ ?

The local speed ratio is the equivalent concept to the tip speed ratio, but considered at different spanwise positions μ . That is:

$$\lambda_r = \mu \lambda = \mu \frac{\Omega R}{u_\infty}.$$

3. What is the effective wind (also called apparent velocity) $\mathbf{u}_a$ at the position μ ?

The apparent wind $\mathbf{u}_a$ is the difference between the freestream wind velocity $\mathbf{u}_\infty$ and the blade's motion $\mathbf{u}_b$ at the station. That is:

$$\mathbf{u}_a = \mathbf{u}_\infty - \mathbf{u}_b.$$

The velocity of the blade points in the tangential direction $\hat{\mathbf{j}}$, with magnitude λu_∞ . That is:

$$\mathbf{u}_b = \mu \Omega R \hat{\mathbf{j}} = \mu \lambda u_\infty \hat{\mathbf{j}}.$$

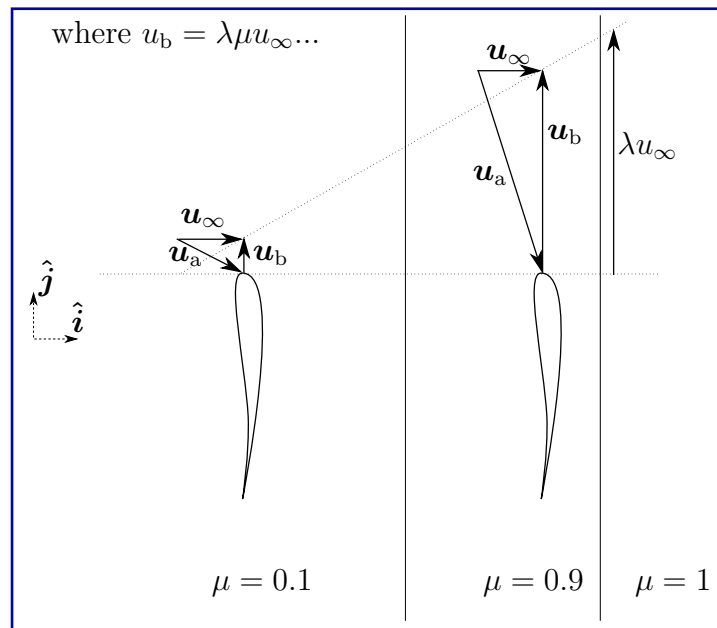
So, the apparent velocity at position μ is:

$$\mathbf{u}_a = u_\infty \hat{\mathbf{i}} - \mu \lambda u_\infty \hat{\mathbf{j}}$$

4. Sketch the velocity triangles for the following positions:

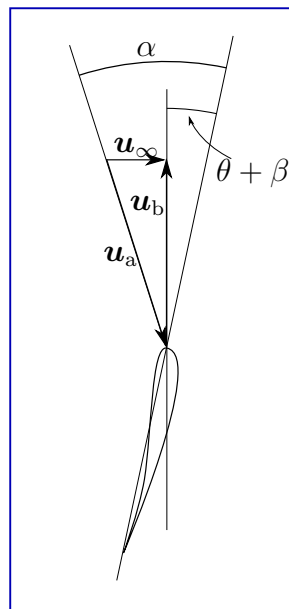
(a) $\mu = 0.1$

(b) $\mu = 0.9$



5. Assume that the blades are uniformly pitched with an angle β , but have a 'perfect' twist distribution $\theta(\mu)$ so that α always takes its design value of 6 degrees if $\beta = 0$. What is $\theta(\mu)$?

The angle of attack is the angle between the chord line and the apparent velocity.



This gives:

$$\alpha = \phi + \theta + \tan^{-1} \left(\frac{1}{\lambda\mu} \right) = \phi + \theta + \cot^{-1}(\lambda\mu).$$

If $\alpha = 6\pi/180$ rad, then:

$$\theta = \frac{1}{30} (-30 \cot^{-1}(\lambda\mu) - 30\phi + \pi)$$

6. For arbitrary lift c_l and drag c_d coefficients, what is the aerodynamic force $d\mathbf{F}_{\text{aero}}$ for an infinitesimal segment of area dA around a position μ ? Assume that the blades point straight, radially outwards.

We know that the aerodynamic force is the sum of the lift and drag forces

$$d\mathbf{F}_{\text{aero}} = d\mathbf{F}_L + d\mathbf{F}_D.$$

By using the definitions of the coefficients, we can see that:

$$d\mathbf{F}_L = c_l \frac{1}{2} \rho \|\mathbf{u}_a\|_2^2 dA \hat{\mathbf{v}}l, \quad d\mathbf{F}_D = c_d \frac{1}{2} \rho \|\mathbf{u}_a\|_2^2 dA \hat{\mathbf{v}}d$$

We know the orientations of these forces because the drag force must be along the apparent velocity, and the lift force must be perpendicular to the drag and the span.

$$\hat{\mathbf{v}}d = \frac{\mathbf{u}_a}{\|\mathbf{u}_a\|_2} = \frac{u_\infty \hat{\mathbf{i}} - \mu \lambda u_\infty \hat{\mathbf{j}}}{\|u_\infty \hat{\mathbf{i}} - \mu \lambda u_\infty \hat{\mathbf{j}}\|_2} = \frac{\hat{\mathbf{i}} - \mu \lambda \hat{\mathbf{j}}}{\sqrt{1 + \mu^2 \lambda^2}}$$

To give a right-handed coordinate system $\hat{\mathbf{k}}, \hat{\mathbf{j}}, \hat{\mathbf{i}}$ in the sketch above: $\hat{\mathbf{k}}$ must point down into the page. Then:

$$\hat{\mathbf{v}}l = \frac{\mathbf{u}_a \times \hat{\mathbf{k}}}{\|\mathbf{u}_a \times \hat{\mathbf{k}}\|_2} = \frac{u_\infty \hat{\mathbf{j}} + \lambda \mu u_\infty \hat{\mathbf{i}}}{\|u_\infty \hat{\mathbf{j}} + \lambda \mu u_\infty \hat{\mathbf{i}}\|_2} = \frac{\hat{\mathbf{j}} + \mu \lambda \hat{\mathbf{i}}}{\sqrt{1 + \mu^2 \lambda^2}}$$

Now we can put all of these expressions together:

$$d\mathbf{F}_{\text{aero}} = \frac{1}{2} \rho u_\infty^2 (1 + \mu^2 \lambda^2)^{\frac{1}{2}} \left(c_l (\hat{\mathbf{j}} + \mu \lambda \hat{\mathbf{i}}) + c_d (\hat{\mathbf{i}} - \mu \lambda \hat{\mathbf{j}}) \right) dA$$

7. What is the mechanical power production $dP(\mu)$ of that segment around position μ ?

The power is the force acting parallel to the blade's motion:

$$dP = d\mathbf{F}_{\text{aero}} \cdot \mathbf{u}_b.$$

Since we know that the blade's motion is in the $\hat{\mathbf{j}}$ direction, we can use the above force expression:

$$dP = \frac{1}{2} \rho u_\infty^2 (1 + \mu^2 \lambda^2)^{\frac{1}{2}} dA (c_l - c_d \mu \lambda) (\lambda \mu u_\infty) = \frac{1}{2} \rho u_\infty^3 \lambda \mu (1 + \mu^2 \lambda^2)^{\frac{1}{2}} dA (c_l - c_d \mu \lambda).$$

Just for abbreviation, let's define $\xi_n := c_l - c_d \mu \lambda$.

8. If the lift c_l and drag c_d coefficients can be found with the following relations, what is the power harvested by the blade segment around position μ ?

$$c_l(\mu) = 1.2\mu, \quad \frac{c_l}{c_d}(\mu) = 100\mu$$

Let's start with the dP expression from above:

$$dP = \frac{1}{2} \rho u_{\infty}^3 \lambda \mu (1 + \mu^2 \lambda^2)^{\frac{1}{2}} dA (c_l - c_d \mu \lambda).$$

If we plug the above lift and drag ($c_d = c_l / (c_l / c_d)$) expressions into this power statement then, it gives the following:

$$dP = \frac{1}{2} \rho u_{\infty}^3 \lambda \mu (1 + \mu^2 \lambda^2)^{\frac{1}{2}} dA \left(1.2\mu - \frac{1.2}{100} \mu \lambda \right).$$

9. What is the relationship between the power harvesting factor ζ and μ ?

The power harvesting factor is the harvested power divided by the power density and the segment area. That is:

$$\zeta = \frac{dP}{\frac{1}{2} \rho u_{\infty}^3 dA} = \lambda \mu (1 + \mu^2 \lambda^2)^{\frac{1}{2}} \xi_n$$

10. How would you go about finding the total power P harvested by the entire turbine? (*Hint: just give the procedure; don't follow it yet.*)

Everything in our problem so far has been symmetrical. That means that the total power must be the sum of all segment powers for all blades:

$$P = B \int_{\mu=0}^{\mu=1} dP$$

Remember that $dP = dP(d\mu)$. So, we'll have to integrate over μ .

11. How would you go about finding the power coefficient C_P of the entire turbine? Use the following definition: $dA = c(\mu) d\mu R$, where $c(\mu)$ is a chord length as a function of μ .

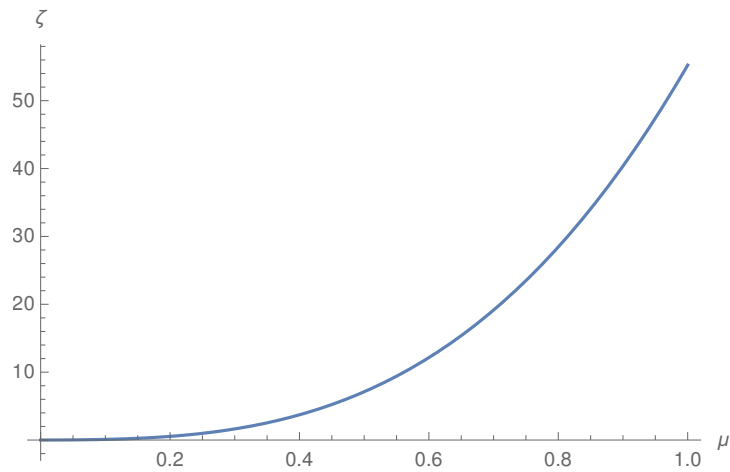
The power coefficient of the entire turbine is the total harvested power divided by the power density (at hub-height, though this is not relevant in a uniform wind field) and the total rotor area πR^2 . That gives:

$$C_P = \frac{P}{\frac{1}{2} \rho u_{\infty}^3 \pi R^2} = \frac{B}{\pi R} \int_{\mu=0}^{\mu=1} \lambda \mu (1 + \mu^2 \lambda^2)^{\frac{1}{2}} \xi_n c(\mu) d\mu$$

12. If we use the above model that we've described to this point, for some given parameter values ($\lambda = 7$, $c_0 = 0.15R$, $c_1 = 0.05R$, $u_{\infty} = 10$ m/s, $\rho = 1.225$ kg/m³, $R = 50$ m and $B = 3$), can you find how much power the full turbine will extract?

*here assume that the chord is a linear interpolation between the chord c_1 at the tip and the chord c_0 at the root: $c(\mu) = c_0 + (c_1 - c_0)\mu$, what gives us $dA = (c_0 + (c_1 - c_0)\mu) d\mu R$

(a) plot the power harvesting factor ζ vs. μ



(b) find how much power the full turbine will extract

When we plug in our values into the dP expression, we get the following ugly numeric expression:

$$dP \approx 1.2 \times 10^6 \mu^2 (1.5 - \mu) \sqrt{49\mu^2 + 1}$$

We can integrate this expression numerically between $\mu = 0$ and $\mu = 1$ to get:

$$P = B \int_0^1 dP \approx 4.5 \cdot 10^6 \text{ W} = 4.5 \text{ MW}$$